Fast Tridiagonal Solvers on GPU

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GPU Technology Conference 2009
Outline

• Introduction
• Algorithms
  – Design algorithms for GPU architecture
• Performance
  – Bottleneck-based vs. Component-based performance model
• Summary
What is a tridiagonal system?

\[
\begin{pmatrix}
  b_1 & c_1 & & \\
  a_2 & b_2 & c_2 & \\
  & a_3 & b_3 & c_3 \\
  & & \ddots & \ddots \\
  & & & a_n & c_{n-1} \\
  & & & & a_n & b_n
\end{pmatrix}
\begin{pmatrix}
  x_1 \\
  x_2 \\
  \vdots \\
  \vdots \\
  x_n
\end{pmatrix}
= 
\begin{pmatrix}
  d_1 \\
  d_2 \\
  \vdots \\
  \vdots \\
  d_n
\end{pmatrix}
\]
What is it used for?

- Scientific and engineering computing
  - Alternating direction implicit (ADI) methods
  - Numerical ocean models
  - Semi-coarsening for multi-grid solvers
  - Spectral Poisson Solvers
  - Cubic Spline Approximation

- Video games and computer-animated films
  - Depth of field blurs
  - Fluid simulation
A Classic Serial Algorithm

• Gaussian elimination in tridiagonal case (Thomas algorithm)

\[
\begin{pmatrix}
1 & c'_1 & c'_2 & c'_3 & c'_4 \\
0 & 1 & c'_2 & c'_3 & c'_4 \\
0 & 0 & 1 & c'_3 & c'_4 \\
0 & 0 & 0 & 1 & c'_4 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5
\end{pmatrix}
=
\begin{pmatrix}
d'_1 \\
d'_2 \\
d'_3 \\
d'_4 \\
d'_5
\end{pmatrix}
\]

Phase 2: Backward Substitution
Parallel Algorithms

• Coarse-grained algorithms (multi-core CPU)
  – Two-way Gaussian elimination
  – Sub-structuring method

• Fine-grained algorithms (many-core GPU)
  – Cyclic Reduction (CR)
  – Parallel Cyclic Reduction (PCR)
  – Recursive Doubling (RD)
  – Hybrid CR-PCR algorithm

A set of equations mapped to one thread
A single equation mapped to one thread
A little history

• Parallel tridiagonal solvers since 1960s:
  – Vector machines: Illiac IV, CDC STAR-100, and Cray-1
  – Message passing architectures: Intel iPSC, and Cray T3E
  – And GPU as well!
Two Applications on GPU

Depth of field blur, Michael Kass et al.
OpenGL and Shader language
Cyclic reduction
2006

Shallow water simulation
CUDA
Cyclic reduction
2007
Cyclic Reduction (CR)

8-unknown system
4-unknown system
2-unknown system
Solve 2 unknowns
Solve the rest 2 unknowns
Solve the rest 4 unknowns

2^\log_2 (8) - 1 = 2^3 - 1 = 5 steps
Parallel Cyclic Reduction (PCR)

4 threads working

One 8-unknown system

Two 4-unknown systems

Four 2-unknown systems

Solve all unknowns

\[ \log_2 (8) = 3 \text{ steps} \]
Hybrid Algorithm (1)

• CR
  – Every step we reduce the system size by half (Good)
  – Some processing cores stay idle if the system size is smaller than the number of cores (Bad)
  – Needs more steps to finish (Bad)

• PCR
  – Fewer steps required (Good)
  – Same amount of work for all steps (Bad)
Hybrid Algorithm (2)

System size reduced at the beginning
No idle processors
Fewer algorithmic steps

Even more beneficial because of:
bank conflicts
control overhead

Switch to PCR
Switch back to CR
GPU Implementation (1)

• Linear systems mapped to multiprocessors (blocks)

• Equations mapped to processors (threads)
GPU Implementation (2)

- Storage need: 5 arrays = 3 diagonals + 1 solution vector + 1 right hand side
- All data resides in shared memory if it fits
- Use contiguously ordered threads to avoid unnecessary divergent branches
- In-place data storage
  - Efficient, but introduce bank conflicts to CR
Performance Results – Test Platform

- 2.5 GHz Intel Core 2 Q9300 quad-core CPU
- GTX 280 graphics card with 1 GB video memory
- CUDA 2.0
- CentOS 5 Linux operating system
Performance Results

Solve 512 systems of 512 unknowns

PCI-E: CPU-GPU data transfer
MT GE: multi-threaded CPU Gaussian Elimination
GEP: CPU Gaussian Elimination with pivoting (from LAPACK)
Performance Analysis

• Factors that determine performance
  – Global/shared memory accesses
  – Bank conflicts
  – Computational complexity
  – Overhead for synchronization and loop control
Bottleneck vs. Pie slice

Performance = \min(\text{factor1, factor2, } \ldots) \quad \text{Performance} = \sum(\text{factor1, factor2, } \ldots)
Performance Measure

A manual differential method

```c
__global__ void scan_naive(float *g_odata, float *g_idata, int n)
{
    // Dynamically allocated shared memory for scan kernels
    extern __shared__ float temp[];

    int thid = threadIdx.x;
    int pout = 0;
    int pin = 1;

    // Cache the computational window in shared memory
    temp[pout*n + thid] = (thid > 0) ? g_idata[thid-1] : 0;
    for (int offset = 1; offset < n; offset *= 2)
    {
        pout = 1 - pout;
        pin = 1 - pout;
        __syncThreads();
        temp[pout*n+thid] = temp[pin*n+thid];
        if (thid >= offset)
            temp[pout*n+thid] += temp[pin*n+thid - offset];
    }
    __syncThreads();

    g_odata[thid] = temp[pout*n+thid];
}
```
Control Overhead

Time doesn't decrease anymore, because GPU vector length is 32 (warp size)

Control and synchronization overhead

A step is very expensive in terms of control

Enforce a stride of one to avoid bank conflicts
Bank Conflicts

There are 16 banks, so all memory accesses are serialized.
CR vs. PCR (1)

Solve 512 systems of 512 unknowns
(Time Breakdown)

- Forward Reduction
- Backward Substitution
- Solve 2-unknown systems
- Global memory

PCR: 0.53 ms
CR: 1.07 ms
CR vs. PCR (2)

**CR**
- Global: 0.27
- Shared: 0.69
- Computation: 0.1
- 26% 9%
- 16 GFLOPS
- 49 GB/s

**PCR**
- Global: 0.11
- Shared: 0.16
- Computation: 0.27
- 20% 30%
- 883 GB/s

Pros: O(n)
Cons: more steps (control overhead), bank conflicts

Pros: fewer steps, no bank conflicts
Cons: O(nlogn)
Pitfalls

• The higher computation rate and sustained bandwidth, the better
  – They may have different algorithm complexity

• The lower algorithm complexity, the better
  – What if there is considerable amount of control overhead, or bank conflicts, or low hardware utilization
PCR vs Hybrid

• Make tradeoffs between the computation, memory access, and control
  – The earlier you switch from CR to PCR
    • The fewer bank conflicts, the fewer algorithmic steps
    • But more work
Hybrid Solver – Sweet Point

Optimal performance of hybrid solver
Solving 512 systems of 512 unknowns

Time (milliseconds)

Optimal intermediate system size
Known issues and future research

- PCI-E data transfer
- Double precision
- Pivoting
- Block tridiagonal systems
- Handle large systems that cannot fit into shared memory
- Automatic performance profiling
Summary

• We studied the performance of 3 parallel solvers on GPU

• We learned two major lessons
  – Component-based rather than bottleneck-based
    » Performance is more complicated than either compute-bound or memory-bound
  – We can make tradeoffs between these components, and we need to make the right tradeoff
Questions?

Thanks